Ergodicity for singular SDEs driven by fBm

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Setting

$(W_t^H)_{t\geq 0}$ is a Gaussian process such that:

$$\mathbb{E}\left[W_t^H W_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

Let $H \in \left(0, \frac{1}{2}\right)$ and consider:

$$dX_t = g(X_t)dt + u(X_t)dt + dW_t^H \tag{1}$$

1. $g \in \mathcal{C}^\alpha$ for $\alpha > 1 - \frac{1}{2H}$ (in particular $\alpha < 0$)
2. confining potential u, $(u(x) - u(y), x - y) \leq -\lambda|x-y|^2$

Problems: well-posedness, invariant measure, uniform in time propagation of chaos

Reminder : $g \in \mathcal{C}^\alpha$ scales

$$\forall \lambda > 0 \;\; g(\lambda x) \simeq \lambda^\alpha g(x)$$

Ergodic theory for singular SDEs

$$X_t = x_0 + \int_0^t g(X_s)ds - \int_0^t X_s ds + W_t$$

$\mathsf{g} \in \mathsf{W}^{1,\infty}(\mathbb{R}^d) \cap \mathsf{L}^\infty(\mathbb{R}^d))$: Cauchy-Lipschitz + coupling

$\mathsf{g} \in \mathsf{L}^q(\mathsf{B}_R(0)) \cap \mathsf{L}^\infty(\mathsf{B}_R(0)^c)$**:** For $q > d$, strong well-posedness by Krylov–Röckner, uniqueness of IM by PDE analysis (Zhang-Zhao '18)
$\boldsymbol{g(x) = \frac{1}{x}\frac{\beta-1}{2}}$**:** Well-posedness on $\mathbb{R}$ for $\beta \in \mathbb{R}$ (confined Bessel) mutually singular invariant measures for $\beta \leq 0$ and $\beta \geq 2$.

$\mathsf{g}(\mathsf{x}) = \nabla\left(|\mathsf{x}|^{-\mathsf{p}+1}\right)$**:** Conditional well-posedness for $d-1 \leq p < d+1$ by reflective singularity, Gibbs measures (Serfaty, Duerinckx and others)

our case - only regularity counts

Our result

$$dx_t = g(x_t)dt + u(x_t)dt + dW_t^H \tag{2}$$

Theorem (M.-Mayorcas '25+)

Let $H \in (0, \frac{1}{2}), g \in \mathcal{C}^\alpha, \alpha > 1 - \frac{1}{2H}$, let u be a confining potential satisfying

$$(u(x) - u(y), x - y) \leq -|x-y|^2$$

. Then:

1. *The equation (2) is well-posed for all $t \geq 0$.*
2. *There exists exactly one invariant measure of X_t*
3. *X has Gaussian tails, i.e. there exists $\kappa_0 > 0$ such that:*

$$\forall \kappa < \kappa_0 \quad \sup_{t\geq 0} \mathbb{E}\left[\exp\left(\kappa\left(|X_t|^2 + \|X\|^2_{\mathcal{C}^{H-\delta};[t,t+1]}\right)\right)\right] \lesssim 1$$

Remark - if $H < 1/4$, g can be Dirac delta.

Regularisation by noise

Solution on finite interval $[0, T]$, $T > 0$:

Theorem (Catellier-Gubinelli 16', Galeati-Gubinelli 22' and many others)

If $g \in \mathcal{C}^{\alpha}, \alpha > 1 - \frac{1}{2H}$, *then*

$$dx_t = g(x_t)dt + dW_t^H$$

has a path-by-path unique solution.

Tools: sewing lemmas, regularising properties of Gaussian r.v., Young integration, Girsanov theorem

Key idea, let $E = \mathcal{C}^{\beta} L_{\omega}^{m}$, $\beta > 1/2, m \geq 2$:

$$\left\| \int_0^{\cdot} g(W_r^H + \varphi_r)dr - \int_0^{\cdot} g(W_r^H + \psi_r)dr \right\|_E \leq C(\|g\|_{\mathcal{C}^{\alpha}}) \|\varphi - \psi\|_E$$

Ornstein-Uhlenbeck process

Let W^H be a standard fractional Brownian motion, ψ a deterministic path and define Y_t as:

$$Y_t^{\psi} = Y_0 + \int_0^t b(Y_r)dr + W_t^H + \psi_t \quad b(x) = -\lambda x$$

$$Y_t^0 \sim_{\mathbb{P}} \mathcal{N}\left(Y_0 e^{-\lambda t}, C_{H,t}\right) \quad C_{H,t} \to_{t\to\infty} C_{H,\infty}$$

Three key properties:

1. $\sup_{t>0} \|Y_t\|_{L^m_\omega} \lesssim \sqrt{m}$ (tightness)
2. Law continuous in starting point and ψ (strong Feller)
3. $\forall A \in \mathcal{B}(\mathbb{R}^d)$ $\mathbb{P}(Y_t \in A) > 0$ (irreducibility)

Nonlinear b - mimic 1),2),3) $\Rightarrow \exists!$ inv. measure (Hairer-Ohashi '07)
+ a technical property of quasi-Markovianity

Tightness - random ODE approach for $g \in \mathcal{C}_b^1$

$$\rho = x - y \quad d\rho_t = g(\rho_t + y_t)dt + u(\rho_t + y_t)dt + y_t dt$$

ρ is a **random ODE**, not an SDE, so that **we do not need Ito calculus** and we can use standard change of variables

$$\begin{aligned}\frac{d|\rho_t|^2}{dt} &= (\rho_t, g(\rho_t + y_t)) + (\rho_t, u(\rho_t + y_t)) + (\rho_t, y_t) \\ &\leq (\rho_t, g(\rho_t + y_t)) - \lambda|\rho|^2 + (\rho_t, y_t - u(y_t))\end{aligned}$$

Variation of constants:

$$|\rho_t|^2 \leq \int_0^t e^{-\lambda(t-r)} \left(\rho_r, g(\rho_r + y_r)\right) dr + \int_0^t e^{-\lambda(t-r)} \left(\rho_r, |y_r| + |u(y_r)|\right) dr \tag{3}$$

Gaussian tightness

Let $C > 0$. Via a **modification** of a stochastic sewing lemma

$$\sup_{t>0} \left\| \int_0^t e^{-\lambda(t-r)} \left(\rho_r, g(\rho_r + y_r)\right) dr \right\|_{L^m_\omega} \lesssim_{\|g\|_{\mathcal{C}^\alpha}} 2^{-C} \|\rho\|^2_{L^m_\omega} + 2^C \sqrt{m} \|\rho\|_{L^m_\omega}$$

1. constant $\sqrt{m}$ - optimal BDG constant
2. balancing $C > 0$ - modifying the sewing proof

With a good choice of $C > 0$:

$$\|\rho\|^2_{L^\infty L^{2m}_\omega} \leq \sqrt{m} \|\rho\|_{L^\infty L^m_\omega} + \|g\|_{\mathcal{C}^\alpha} + 2^C m + \|Y\|_{L^m_\omega}$$

As a result

$$\|\rho\|_{L^\infty L^m_\omega} \lesssim \sqrt{m}$$

Uniqueness of invariant measure

Let $(x, t, W^H) \to \Phi(x, t, W^H)$ be a solution map, where $x_0 = x$.

1. Strong Feller:

   1.1 we need to get good bounds on the Jacobian:

   $$D_x\Phi(x, t, W^H) = I + \int_0^t (\nabla b)(X_r) D_x\Phi(x, r, W^H)dr$$

   Building up on the tightness results:

   $$\sup_{t>0} \mathbb{E}\left[\exp\left(\kappa \left\| \int_0^\cdot (\nabla g)(X_r)dr \right\|^2_{\mathcal{C}^{\mathbf{1}+H(\alpha-\mathbf{1})-\delta}}\right)\right]$$

   1.2 $\Phi(x, t, W^H + \psi) \to_{\|\psi\|\to 0} \Phi(x, t, W^H)$ if ψ deterministic, proof by interpolation in sewing

2. Irreducibility - by Girsanov

$$\tau \simeq 0 \;\; \mathbb{P}(X_\tau \in A) \simeq \mathbb{P}\left(W^H_\tau \in A\right)$$

Applications? What's next?

1. **Sampling**: numerical studies to investigate the shape of invariant measure.
2. Shape of invariant measure around the origin

$$\mathbb{E}\left[\exp\left(\kappa|X_t|^{\beta}1_{[X_t\neq 0]}\right)\right] \quad \beta < 0$$

   What value of β ?
3. **Interacting particle systems**: Uniform in time propagation of chaos and invariant measure of McKean-Vlasov.

$$dx_t^i = \frac{1}{N}\sum_{j=1,j\neq i}^{N} g(x_t^i - x_t^j)dt + u(x_t^i)dt + \epsilon(N)dW_t^{H,i}$$

   asymptotics for an invariant measure for well-chosen $\epsilon(N)$?