Small noise limit for a singular SDE

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Joint work with Paul Gassiat

Motivating question

Consider a function b and a noise $(W_t)_{t>0}$ such that SDE (1) is well-defined:

$$X_t = X_0 + \int_0^t b(X_r)dr + \epsilon W_t \tag{1}$$

Our problem:

1. What happens if $\epsilon \to 0$?
2. Obvious if b Lipschitz, if not?

Plan of the talk

1. Regularisation by noise and key ideas why does it work
2. Zero noise limit in 1D
3. work in progress, open questions and future work

Peano example

Consider $b(x) = \text{sgn}(x)|x|^\gamma$, $\gamma < 1, \epsilon = 0$. Then:

$$X_t = \int_0^t b(X_r)dr \quad X_t = C_E(t - t_0)_+^{\frac{1}{1-\gamma}}$$

with:

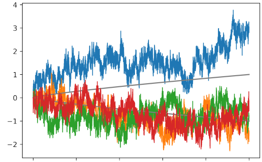
$$C_E = |1-\gamma|^{\frac{1}{1-\gamma}} \quad t_0 \in \mathbb{R}_+$$

Infinite number of solutions.

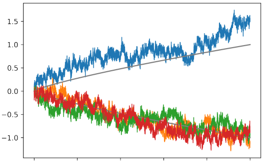
Relevance : fluid dynamics - you tell me !

We will justify these plots...

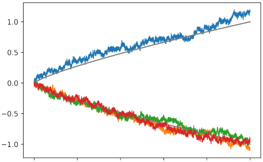
$$dx_t = b(x_t)dt + \epsilon dW_t^H \quad b(x) = \mathrm{sgn}(x)|x|^\gamma, \gamma < 1$$



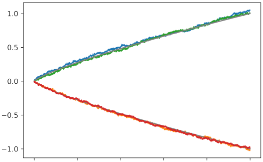
(a) $\epsilon = 1$



(b) $\epsilon = 0.3$



(c) $\epsilon = 0.1$



(d) $\epsilon = 0.03$

Fractional Brownian motion

W^H is a centered Gaussian process:

$$\mathbb{E}\left(W_t^H - W_s^H\right)^2 = (t-s)^{2H}$$

- NOT a martingale
- NOT a Markov process

$$W_t^H = \int_{-\infty}^t k(t,r)dB_r$$

Lower $H \Rightarrow$ better regularisation

Well-posedness with $\epsilon > 0$

Different contexts

1. Zvonkin 74', Veretennikov 81', Davie 08', W Brownian motion, $b \in L^\infty$
2. Catellier-Gubinelli 16', W fractional Brownian motion. $b \in \mathcal{C}^\gamma$ (Hölder continuous functions).

$$\gamma > 1 - \frac{1}{2H} \quad W = W^H, \mathbb{E}\left(W_t^H - W_s^H\right)^2 = |t-s|^{2H}$$

3. Krylov-Röckner 05', W Brownian motion, $b \in L_q L_p$ for $\frac{1}{q} + \frac{d}{p} < 1$, strong solutions (different scheme of proof)

Well-posedness with $\epsilon > 0$

$$X_t = X_0 + \int_0^t b(X_r)dr + W_t$$

Let X_t, Y_t be two solutions with $X_0 = Y_0$ and $W = 0$

$$|X_t - Y_t| \leq \|\nabla b\|_{L^\infty} \int_0^t |X_r - Y_r|dr$$

so:

$$\|X - Y\|_\infty \leq \|\nabla b\|_{L^\infty} \|X - Y\|_\infty \, t$$

Can we find a substitute for Lipschitz condition in the presence of the noise W_t? If so:

$$\|X - Y\| \leq \|b\|_{\mathcal{C}^\gamma} \|X - Y\| \, t^\alpha$$

$\gamma < 1, \alpha > 0$, then $\|b\|_{\mathcal{C}^\gamma} t^\alpha < 1/2$

Well-posedness with $\epsilon > 0$

We change variables:

$$\theta^X = X_t - W_t = \int_0^t b(\theta_r^X + W_r)dr$$

Can we justify ?:

$$|\int_0^t b(W_r+\theta_r^X)-b(W_r+\theta_r^Y)dr| \leq \left\|\nabla \int_0^\cdot b(W_r + \cdot)dr\right\|_\infty \left\|\theta^X - \theta^Y\right\|$$

Change point of view:

$$\underbrace{x \mapsto b(x)}_{\text{irregular}} \quad \underbrace{x \mapsto \int_0^t b(W_r + x)dr}_{\text{smooth}}$$

How?

Main tools:

1. stochastic sewing (Le 18, Delarue-Diel 16):

$$\int_0^{\cdot} (\nabla b)(W_r^H + x)dr =_{L^p} \lim_{|t_{i+1}-t_i|\to 0} A_{st}(x)$$

   where A_{st} a local approximation

2. nonlinear Young equations (Catellier-Gubinelli 16, Galeati 22):

$$\theta_t = \int_0^t A(dr, \theta_r) \quad A \in \mathcal{C}^\beta \mathcal{C}^1, \beta > 1/2$$

3. convolutions with Gaussian density:

$$g \in L^p \ \ \|P_{t^{2H}}(\nabla g)\|_{L^\infty} \lesssim t^{1+H(\frac{1}{p}-1)} \|g\|_{L^p}$$

Power counting heuristic

$$X_t = x_0 + \int_0^t b(X_r)dr + W_t^H$$

1. Let $b \in \mathcal{C}^\gamma \cap \mathcal{C}^1(\mathbb{R} \setminus \{0\}), \gamma < 1$ - one singularity at zero, ex.:

$$b(x) = \mathrm{sgn}(x)|x|^\gamma$$

2. Local (!) uniqueness almost everywhere except $x_0 = 0$.
3. If $W^H \in \mathcal{C}^{H-}$ then $X \in \mathcal{C}^{H-}$.
4. $b(X) \in \mathcal{C}^{\gamma H}$ and then $\int_0^\cdot b(X_r)dr \in \mathcal{C}^{1+\gamma H}$
5. As $t^H > t^{1+\gamma H}$ we have:

$$W_t^H \gg \int_0^t b(X_r)dr$$

6. we escape the bad point

Did we just cheat?

Not a lot !

Proposition (Galeati-Harang-Mayorcas 22')

For X_t solution of:

$$X_t = X_0 + \int_0^t b(X_r)dr + W_t^H$$

there holds:

$$\sup_{t<T} t^{-(1+H\gamma)} \left| \int_0^t b(X_r)dr \right| < K \quad \mathbb{E} \exp\left(K^2\right) < \infty$$

Proof.

Girsanov + similar estimate for $\int_0^t b(W_r^H)dr$ via stochastic sewing. $\square$

Back to Peano example - limit of $\epsilon \to 0$

$$x_t = x_0 + \int_0^t \mathrm{sgn}(x_r)|x_r|^\gamma dr + \epsilon W_t$$

$$y_t^{\pm} = \pm C_E t^{\frac{1}{1-\gamma}} \quad C_E = \left(\frac{1}{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

Existing results:

1. Bafico-Baldi 82' - $X_t^\epsilon \to y_t^{\pm}$ with PDE methods, more general b:

$$b(x)\mathrm{sgn}(x) \geq 0$$

2. Gradinaru-Herrmann-Roynette 01' - Ito calculus $\Rightarrow$ exponential rate of convergence
3. Delarue-Flandoli 14' - dynamical proof, without rate of convergence
4. Pilipenko-Proske 18' - convergence for $H \neq 1/2, \gamma > 0$ (not quantified)

Transition point

$$X_t^\epsilon = \int_0^t \operatorname{sgn}(X_r)|X_r|^\gamma dr + \epsilon W_t^H \tag{2}$$

Heuristic:

1. Initially $X_t \simeq \epsilon W_t^H$, for small t
2. $$\epsilon W_t^H \simeq \epsilon t^H \quad \int_0^t \operatorname{sgn}(X_r)|X_r|^\gamma dr \simeq \epsilon^\gamma t^{1+H\gamma}$$
3. Noise dominates $t \simeq \epsilon^{\frac{1-\gamma}{1+H(\gamma-1)}} =: t_\epsilon$...
4. ...and drags the path away from zero, above $x_\epsilon = t_\epsilon^{\frac{1}{1-\gamma}}$

Gaussian scaling - $X_\cdot^\epsilon =_{\mathbb{P}} x_\epsilon X^1_{\cdot/t_\epsilon}$

Theorem (M.-Gassiat 24+)

Let $\gamma > 1 - \frac{1}{2H}$. There exists the random variable ρ such that for some $\kappa \in (0,1)$:

$$\forall s \in [0,1] \;\; |X^{\epsilon}_{t_\epsilon \rho + s}| \to C_E s^{\frac{1}{1-\gamma}} = y^{+}_{s} \qquad \ln \mathbb{P}(\rho > t) \lesssim -t^{\kappa}$$

Moreover:

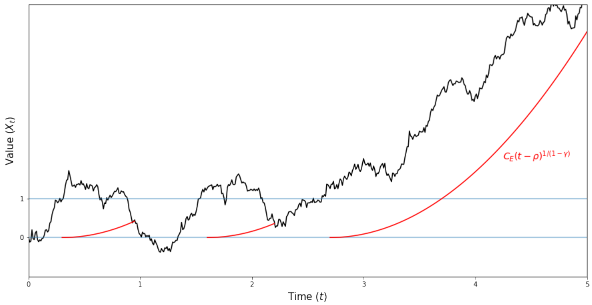
$$|X^{\epsilon}_{t_\epsilon \rho + s} - \pm y^{+}_{t_\epsilon + s}| < \epsilon^{\alpha} \qquad \alpha \in (0,1)$$

and sign is chosen at random.

κ can be computed - in the Brownian case ($H = 1/2$) and $\gamma > 0$ there holds $\kappa = 1$.

arXiv - "Zero noise limit for singular ODE regularized by fractional noise", Gassiat, P., Mądry, Ł.

Ergodic approach



1. W Brownian motion - strong Markov property $\Rightarrow$ we can restart the dynamics
2. W^H fractional Brownian motion - no SMP, but:

$$\forall |t-s| > 1 \ \ |\mathbb{E}\left[W^H_{s,s+1} W^H_{t,t+1}\right]| \lesssim |t-s|^{2H-2}$$

Ingredients of the solution

1. Knowledge of the exact solution -

$$\epsilon = 0 \Rightarrow X_t = C_E t^{\frac{1}{1-\gamma}}$$

2. one-dimensionality of the problem
3. Mandelbrot representation with $(B_t)_{t \in \mathbb{R}}$:

$$W_t^H = \int_{-\infty}^{t} (t-r)_+^{H-1/2} - (-r)_+^{H-1/2} dB_r \quad (\cdot)_+ = \max(\cdot, 0)$$

point 3 is critical - quantification of the influence of the past !

Markovianization (Stage 1)

Problem - let $t, s > 0$. $\int_{-\infty}^{s} (t-r)^{H-1/2} - (s-r)^{H-1/2} dB_r$ depends on the whole past !

Solution:

Find a sequence of times $(\tau_k)_{k \geq 1}$

$$W_t^H - W_{\tau_k}^H = \int_{\tau_k}^{\tau_k + t} (\tau_k + t - r)^{H-1/2} dB_r + \underbrace{u_t - u_{\tau_k}}_{\text{small}} \quad \|u_{\tau_k + \cdot}\|_{\mathcal{C}^H} < 1$$

tails:

$$\mathbb{E}\left[\exp\left(b\tau_k^{\alpha}\right)\right] < (1+b)^k \quad b > 0, \alpha \in (0,1)$$

Ideas from the study of ergodic properties of fractional SDEs - Hairer 05', Panloup-Richard 20'

Markovianization

Let $G(t,s,r) = (t-r)^{H-1/2} - (s-r)^{H-1/2}$ and

$$W_t^H - W_s^H = \int_s^t (t-r)^{H-1/2} dB_r + \int_{-\infty}^s G(t,s,r) dB_r$$

Remote and recent past:

$$\int_{-\infty}^{s-\Delta} G(t,s,r) dB_r + \int_{s-\Delta}^s G(t,s,r) dB_r = I + J$$

For J:

1. fractional integration*:

$$J \lesssim \|B\|_{[s-\Delta,s]}$$

2. $$c > 0 \;\; \mathbb{P}^{\mathcal{F}_{s-\Delta}}(J < c) > \lambda > 0$$

*Picard 11, "Representation formulae for fBm"

Markovianization - forgetting property

Let $k \in \mathbb{N}, z^k < w^k$, for $[z^k, w^k]$ failed attempt and use integration by parts for fractional integrals*:

$$\sup_{u>0} u^{-1} \left| \int_{z^k}^{w^k} G(u+v, v, r) dB_r \right| \lesssim |v - w^k|^{H-1} \underbrace{\left(\sup_{s \in [z^k, w^k]} \frac{|B_{w^k} - B_s|}{(1 + w^k - s)^{1/2+\delta}} \right)}_{L(z^k, w^k)}$$

Takeaway - for any interval $[z^k, w^k]$ we can wait until rhs is small (integrable):

$$v - w^k > k^{\mu_1} + L(z^k, w^k)^{\mu_2} \quad \mu_1, \mu_2 \geq 1$$

*Picard 11, "Representation formulae for fBm" or Stage 1 in MG24 "Zero noise..."

Leaving critical strip (Stage 2)

After Markovianization:

$$X_t = \int_0^t b(X_r)dr + \int_0^t (t-r)^{H-1/2} dB_r + u_t$$

Recall (Galeati-Harang-Mayorcas 22', Prop. 3.8):

$$\sup_{t<1} t^{-(1+H\gamma)} \left| \int_0^t b(X_r)dr \right| < K \quad \mathbb{E}^{\mathcal{F}_0} \exp\left(K^2\right) < \infty$$

For any *fixed* t_e there exists $\lambda > 0$:

$$\mathbb{P}^{\mathcal{F}_0}\left(\int_0^{t_e} (t_e - r)^{H-1/2} dB_r > 3t_e^H\right) = \mathbb{P}\left(\mathcal{N}(0,1) > 3\right) > \lambda > 0$$

Leaving critical strip

Fix t_e small and let A be the event such that:

1. $\int_0^{t_e}(t_e - r)^{H-1/2}dB_r > 3t_e^H$
2. $|\int_0^{t_e} b(X_r)dr| < Kt_e^{1+H\gamma} < t_e^H \Rightarrow K < t_e^{-(1+H(\gamma-1))}$

$$\mathbb{P}^{\mathcal{F}_0}(A) > \lambda > 0$$

think about power counting ! Result:

$$X_{t_e} > \underbrace{3t_e^H}_{\int_0^\cdot(\cdot - r)^{H-1/2}dB_r} - \underbrace{t_e^H}_{\int_0^\cdot b(X_r)dr} - \underbrace{t_e^H}_{u.} > t_e^H$$

More technicalities - $\mathbb{P}^{\mathcal{F}_0}(X_1 > 1) > \lambda > 0$

Drift wins with the noise (Stage 3)

Let $A > 0, X_0 > 1$ and $b(x) = \mathrm{sgn}(x)|x|^\gamma$.

$$X_t = X_0 + A\int_0^t b(X_r)dr + w_t \quad \varphi(x_0, t) = x_0 + \int_0^t b(\varphi(x_0, r))dr$$

Lemma

Let $(w_t)_{t\geq 0}$ be a continuous path such that for some $\beta > 0$:

$$\sup_{s<t} \frac{|w_t - w_s|}{(1+t-s)^\beta} \leq A^\beta X_0$$

Then:

$$\forall t > 0 \;\; X_t > \varphi(X_0, t) - c(1+t)^\beta \quad c \in (0,1)$$

Proof.

b is Lipschitz away from zero, so we can use comparison principles and exact flow expressions $\square$

Drift wins with the noise - local approximation

$$X_t = \theta_t + w_t = \int_0^t b(X_r)dr + w_t = \int_0^t b(\theta_r + w_r)dr + w_t$$

Let $\bar{w}_T = \sup_{r<T}|w_r|$ and for $b \in \mathcal{C}^1, \partial_x b < 0$

$$\theta_t > \theta_0 + \int_0^t b(\theta_r + \bar{w}_T)dr$$

Therefore:

$$\theta_t > \varphi(\theta_0 + \bar{w}, t) - \bar{w}_T$$

And:

$$X_t > \underbrace{\varphi(X_0 + \bar{w}, t)}_{\simeq T^{\frac{1}{1-\gamma}}} - \underbrace{2\bar{w}_T}_{\simeq(1+T)^{H+\delta}}$$

Drift wins with the noise - condition check

We need:

$$\sup_{s<t} \frac{|w_t - w_s|}{(1+t-s)^\beta} \leq A^\beta X_0$$

In our case:

$$w_t - w_s = \underbrace{\int_s^t (t-r)^{H-1/2} dB_r + \int_0^s G(t,s,r) dB_r}_{I(t,s)} + \underbrace{\int_{-\infty}^0 G(t,s,r) dB_r}_{J(t,s)}$$

1. $\forall s < t \; |J(t,s)| \leq (1+t-s)^{H+\delta}$ a.s.
2. $$\mathbb{P}^{\mathcal{F}_0}\left(\forall s < t \;\; |I(t,s)| \leq (1+t-s)^{H+\delta}\right) > \lambda > 0$$

Result - we can apply Lemma from previous slide.

Construction of ρ from the main theorem

Recall that ρ is a random time such that:

$$\forall t > 0 \; X_{\rho t_\epsilon + t} > \varphi(x_\epsilon, t) - t^{H+\delta}$$

Therefore:

$$\rho_k = \sum_{j=0}^{k} \Delta_j^1 + \Delta_j^2 + \Delta_j^3$$

1. Δ_j^1 - "neutralize" the past noise
2. Δ_j^2 - escape from $[-1, 1]$
3. Δ_j^3 - stay above the solution

$$\rho = \rho_{k^*} \quad k^* = \inf\{k \in \mathbb{N} : \Delta_{k+1}^3 = \infty\}$$

Open questions

1. Optimal rate ? "True" rate probably $\kappa = 1$ for all $H \in (0, 1)$
2. Probability of choice of each trajectory ? Far from clear

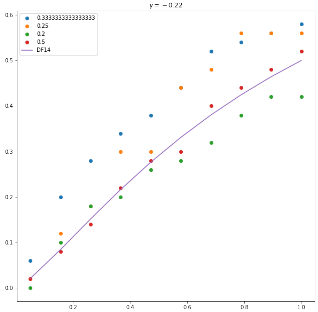


Figure: $A^+/A^-, \gamma = -.2, \epsilon = 1, T = 10000$

Future work

1. Multidimensional case - work in progress (some partial progress, many more challenges)
2. Distribution-dependent SDEs ?
3. Vortex systems - burst and collapse:

$$dx_t^i = \sum_{j \neq i}^{N} b(x_t^i - x_t^j)dt + \epsilon dw_t^i \quad b(x) = x^{\perp}/|x|^2$$

   Numerical results - Grotto, Romito, Viviani 23' for $N = 3$. Rigorous proof missing.
4. Ergodicity - $b(x) = g(x) - x^{\alpha}, \alpha \geq 1$, g repulsive and singular. Large time behaviour, uniqueness and existence of invariant measure

Thank you for your attention !